Name: _____

- This packet is due at **12:00 pm (noon) PDT on Friday, June 4**.
- Neatly fill out this packet either directly or on your own paper. Then upload your answers to Canvas. Please clearly number each problem and keep them in order.
- For each problem in this packet, **organize and show all of your work and justification** for your answer.
- Make sure that it is clear what your final answer is. This may mean boxing/circling it or it may mean doing neither.
- If you are required to draw a graph of a function, each axis must be labeled and given a scale.
- I will not be grading every individual problem, but answers will be available on Canvas after the due date.
- I am happy to help you however I can. Here are some options:
  - Come to my office hours (see syllabus).
  - Email me directly at gknapp4@uoregon.edu

1. Find the Maclaurin series for the hyperbolic sine function: $\sinh(x) = \frac{e^x - e^{-x}}{2}$. Fully simplify your answer and be sure to include the interval of convergence.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$\frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \right)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1 + (-1)^n}{n!} \right) x^n$$

$$= \frac{1}{2}\left(\frac{1+1}{0!} + \frac{1-1}{1!}\cdot x + \frac{1+1}{2!}x^2 + \frac{1-1}{3!}x^3 + \cdots \right)$$

$$= \frac{1}{2}\left(\frac{2}{0!} + 0x + \frac{2}{2!}x^2 + 0x^3 + \frac{2}{4!}x^4 + \cdots \right)$$

$$= \frac{1}{0!} + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$$

The Maclaurin series for e^x and e^{-x} converge for all $x \in (-\infty, \infty)$, so the Maclaurin series for $\sinh(x)$ also converges for all $x \in (-\infty, \infty)$

2. Use the identity $2\sin(x)\cos(x) = \sin(2x)$ to find the power series expansion of $\sin^2(x)$ at $x = 0$.

Hint: What is $\int 2\sin(x)\cos(x)\,dx$? (use u-substitution)

$$\int 2\sin(x)\cos(x)\,dx = \int 2u\,du = u^2 + C = \sin^2(x) + C$$

$u = \sin(x)$

$du = \cos(x)\,dx$

$$\Rightarrow \sin^2(x) = \int 2\sin(x)\cos(x)\,dx = \int \sin(2x)\,dx$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{x^{2n+2}}{(2n+2)}$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!}$$

$$0 = \sin^2(0) = C + 0 = C$$

$$\Rightarrow \sin^2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!}$$

3. Find ~~the smallest~~ a value of n such that the remainder estimate $|R_n| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1}$ for the *Maclaurin Series* for $f(x) = \sin(x)$ yields $|R_n| < 10^{-3}$ at $x = 2$. Compute $p_n(2)$ using your favorite computational tool.

Want: $|R_n(2)| < 10^{-3}$

Know: If $|f^{(n+1)}(z)| \leq M$ for $z \in [0,2]$, then $|R_n(2)| \leq \frac{M}{(n+1)!}2^{n+1}$

Note: $f^{(n+1)}(z) = \pm\sin(z)$ or $\pm\cos(z)$, so

$|f^{(n+1)}(z)| \leq 1$ for $z \in [0,2]$

$\Rightarrow |R_n(2)| \leq \frac{1}{(n+1)!}2^{n+1} = \frac{2^{n+1}}{(n+1)!}$

If $\frac{2^{n+1}}{(n+1)!} < 10^{-3}$, then $|R_n(2)| < 10^{-3}$

Trial and error: $n = 9$ works

$\sin(2) \approx \sum_{n=0}^{4} \frac{(-1)^n}{(2n+1)!}2^{2n+1} \approx 0.909$

4. Find a value of k so that the kth Taylor polynomial for $\ln(1-x)$ centered at $a=0$ approximates $\ln(1-x)$ to within 0.001 for every x in the interval $[-0.5, 0.5]$.

Recall: The Maclaurin series for $\ln(1-x)$ is $\sum_{n=1}^{\infty} -\frac{x^n}{n}$

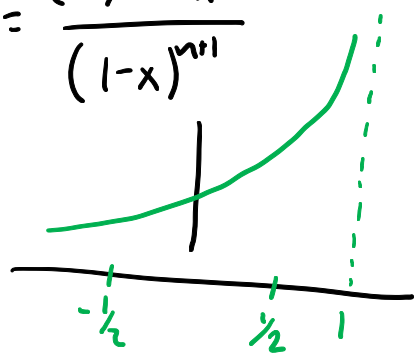
If $|f^{(n+1)}(x)| \le M$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-0|^{n+1} \quad \text{for } x \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

$f'(x) = \frac{-1}{1-x}$, $f''(x) = \frac{1}{(1-x)^2}$, $f'''(x) = \frac{-2}{(1-x)^3}$, $f^{(4)}(x) = \frac{6}{(1-x)^4}$

$f^{(n)}(x) = \frac{(-1)^n (n-1)!}{(1-x)^n} \rightarrow f^{(n+1)}(x) = \frac{(-1)^{n+1} \cdot n!}{(1-x)^{n+1}}$

$\rightarrow |f^{(n+1)}(x)| = \frac{n!}{(1-x)^{n+1}} \rightarrow$



$\rightarrow |f^{(n+1)}(x)| \le \frac{n!}{(1-\frac{1}{2})^{n+1}} = \frac{n!}{(\frac{1}{2})^{n+1}} = 2^{n+1} \cdot n! = M$

$\Rightarrow |R_n(x)| \le \frac{2^{n+1} \cdot n!}{(n+1)!} \cdot |x|^{n+1} \le \frac{2^{n+1} \cdot n!}{(n+1)!} \cdot \frac{1}{2^{n+1}} = \frac{1}{n+1}$

Goal: $\frac{1}{n+1} < 0.001 \longleftrightarrow 1000 < n+1$

Take $n = 1000$

Will NOT be on the final exam

5. This problem will not be graded (so you don't have to submit a solution for this problem). But it will give you an idea of a typical final exam problem about differential equations. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function which satisfies $\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0$ where $y(0) = 1$ and $\frac{dy}{dx}\big|_{x=0} = 0$. Find a formula which relates ~~a_{n+2}, a_{n+1}~~, and ~~a_n~~. Compute ~~$a_0, a_1, \ldots a_5$~~.

c_{n+2}, c_{n+1} c_n $c_0, c_1, \ldots, c_5$

$$y = \sum_{n=0}^{\infty} c_n x^n \rightarrow \boxed{1 = y(0) = c_0}$$

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n \rightarrow \boxed{0 = \frac{dy}{dx}\Big|_{x=0} = c_1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0 \rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\rightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} - (n+1) c_{n+1} + c_n \right] x^n = 0$$

$$\Rightarrow \boxed{(n+2)(n+1) c_{n+2} - (n+1) c_{n+1} + c_n = 0}$$

$$\rightarrow 2 \cdot 1 \cdot c_2 - c_1 + c_0 = 0 \rightarrow 2c_2 + 1 = 0 \rightarrow \boxed{c_2 = -\frac{1}{2}}$$

$$\rightarrow 3 \cdot 2 c_3 - 2c_2 + c_1 = 0 \rightarrow 6c_3 - 1 = 0 \rightarrow \boxed{c_3 = \frac{1}{6}}$$

$$\rightarrow 4 \cdot 3 c_4 - 3c_3 + c_2 = 0 \rightarrow 12c_4 - \frac{1}{2} - \frac{1}{2} = 0 \rightarrow \boxed{c_4 = \frac{1}{12}}$$

$$\rightarrow 5 \cdot 4 c_5 - 4c_4 + c_3 = 0 \rightarrow 20c_5 - \frac{1}{3} + \frac{1}{6} = 0 \rightarrow \boxed{c_5 = \frac{1}{120}}$$